

Retrograde resonance in the planar three-body problem

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Received: date / Accepted: date

Abstract We continue the investigation of the dynamics of retrograde resonances initiated in Morais & Giuppone (2012). After deriving a procedure to deduce the retrograde resonance terms from the standard expansion of the three-dimensional disturbing function, we specialize in the planar problem and construct surfaces of section that explore phase-space in the vicinity of the main retrograde resonances ($2/-1$, $1/-1$ and $1/-2$). In the case of the $1/-1$ resonance for which the standard expansion is not adequate to describe the dynamics, we develop a semi-analytic model based on numerical averaging of the unexpanded disturbing function, and show that the predicted libration modes are in agreement with the behavior seen in the surfaces of section.

Keywords Resonance. Three-body problem. Surface of section.

1 Introduction

The discovery of extrasolar planets that orbit their host stars in the direction opposite the star's rotation has renewed interest in the dynamics of retrograde motion in gravitational systems (Triaud et al., 2010). In the solar system, retrograde motion is confined to smaller bodies such satellites of the outer planets and long period comets. Understanding the structure of retrograde motion and in particular retrograde resonances will help elucidate the origin and evolution of the observed systems. Gayon & Bois (2008) performed numerical integrations of systems with planets moving in opposite directions and observed that a retrograde resonance in two-planet systems is more stable than the equivalent prograde resonance confirming the idea that when two bodies orbit in different directions, encounters occur at a higher relative velocity during a shorter time and mutual perturbations are therefore weaker. In Gayon et al. (2009) the authors obtained the $2/1$ retrograde resonance disturbing function, and identified retrograde resonance angles. However, they were unable to find initial conditions that correspond to libration in retrograde resonance, and did not identify the theoretical

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reason for the observed enhanced stability of retrograde resonances. In Morais & Giuppone (2012) (hereafter Paper I) we compared the stability of prograde and retrograde planets within a binary system. We observed that retrograde planets remain stable nearer to the secondary star than prograde planets. We showed that instability is caused by single mean motion resonances (MMRs) and the possible overlap of adjacent pairs. We used a standard expansion of the disturbing function for the planar circular restricted three-body problem (CR3BP) to obtain the retrograde resonance terms, and we explained how these terms show why retrograde resonances are more stable than prograde resonances as the magnitude of the p/q resonance terms is proportional to a power of the eccentricity, which at the lowest order, is e^{p+q} in retrograde case and e^{p-q} in prograde case.

In this paper, we continue our investigation of retrograde motion in the three-body problem by systematically studying the structure of the phase space near the main retrograde resonances. We concentrate on the planar problem and examine in detail motion near the 2/1, 1/1 and 1/2 retrograde resonances. In section 2, we explain how to obtain the retrograde resonance terms from an expansion in Laplace coefficients of the three-dimensional disturbing function. We also show that, in the planar problem, these coincide with the retrograde resonance terms obtained in Paper I. Since Laplace coefficients diverge when the semi-major axes' ratio is close to unity, in section 3, we develop a semi-analytical model for the co-orbital 1/1 retrograde resonance based on numerical averaging of the unexpanded disturbing function. In section 4, we present our numerical approach and results and describe how retrograde motion phase-space is structured and where stable motion is possible. Section 5 contains a discussion of our results.

2 Differences between prograde and retrograde resonance

The encounter of two bodies in a retrograde configuration (orbiting in opposite directions) occurs at a higher relative velocity during a shorter time than in a prograde configuration. This implies that mutual perturbations are weaker for retrograde MMRs. In Paper I, we studied retrograde MMRs analytically in the context of the planar circular restricted three-body problem and compared their relative strength and stability to prograde resonances. Here, we will explain how we obtain the slow terms of the disturbing function for a p/q retrograde MMR in the three-dimensional CR3BP.

2.1 Disturbing function

Consider a test particle that moves under the gravitational effect of a binary composed of a primary with mass M_* and a secondary with mass $m \ll M_*$. The motion of m with respect to M_* is a circular orbit of radius $a' = 1$ and longitude angle λ' . The reference plane is defined by the binary's orbit. The test particle's osculating Keplerian orbit with respect to M_* has semi-major axis a , eccentricity e , inclination I , true anomaly f , argument of pericentre ω , and longitude of ascending node Ω . The disturbing function reads:

$$R = G m (1/\Delta - r \cos \psi) \quad (1)$$

where r is the radius of test particle, ψ is the angle between the radius vectors of binary and test particle, $\Delta^2 = 1 + r^2 - 2r \cos \psi$, and

$$\cos \psi = \cos(\Omega - \lambda') \cos(\omega + f) - \sin(\Omega - \lambda') \sin(\omega + f) \cos I. \quad (2)$$

The first term of R (direct perturbation) is the gravitational force from the mass m on the test particle whereas the second term (indirect perturbation) comes from the reflex motion of the star under the influence of the mass m as the standard coordinate system is chosen to be centered on the star.

The classic series of the disturbing function is expanded in powers of $\sin^2(I/2)$. This is adequate for nearly coplanar prograde motion since $\sin^2(I/2) \approx 0$ but not for nearly coplanar retrograde motion since $\sin^2(I/2) \approx 1$. Therefore, for nearly coplanar orbits, we define $\beta \ll 1$ such that $I = \beta$ or $I = 180^\circ - \beta$, respectively, for prograde or retrograde motion. We may therefore write:

$$\cos \psi = (1 - s^2) \cos(f + \omega \pm (\Omega - \lambda')) + s^2 \cos(f + \omega \mp (\Omega - \lambda')), \quad (3)$$

where $s^2 = \sin^2(\beta/2) \ll 1$, and the \pm sign applies to prograde or retrograde cases, respectively.

Next, we write $\Delta^2 = 1 + r^2 - 2r \cos(f + \omega \pm (\Omega - \lambda')) - 2r\Psi$, where Ψ is defined as:

$$\Psi = \cos \psi - \cos(f + \omega \pm (\Omega - \lambda')) = 2s^2 \sin(\pm(\Omega - \lambda')) \sin(\omega + f). \quad (4)$$

Expanding the direct perturbation term Δ^{-1} in the vicinity of $\Psi = 0$ (as $s^2 \ll 1$), we may write:

$$\frac{1}{\Delta} = \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2} \left(\frac{1}{2}r\Psi\right)^i \frac{1}{\Delta_0^{2i+1}}, \quad (5)$$

where $\Delta_0^2 = 1 + r^2 - 2r \cos(f + \omega \pm (\Omega - \lambda'))$.

Finally, defining $\epsilon = r/a - 1 = \mathcal{O}(e)$ and expanding $\Delta_0^{-(2i+1)}$ around $\epsilon = 0$:

$$\frac{1}{\Delta_0^{2i+1}} = \left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \epsilon^k \alpha^k \frac{d^k}{d_k \alpha}\right) \frac{1}{\rho^{2i+1}}, \quad (6)$$

with $\alpha = a/a'$, and

$$\begin{aligned} \frac{1}{\rho^{2i+1}} &= (1 + \alpha^2 - 2\alpha \cos(f + \omega \pm (\Omega - \lambda')))^{-(i+1/2)}, \\ &= \sum_j \frac{1}{2} b_{i+1/2}^j(\alpha) \cos(j(f + \omega \pm (\Omega - \lambda'))), \end{aligned} \quad (7)$$

where $b_{i+1/2}^j(\alpha)$ are Laplace coefficients. For $\alpha < 1$, they may be expanded as convergent series in α (Ellis & Murray, 2000).

Examination of the angles in the expression of Ψ (4) and that of ρ (7) show that there are two ways with which the expansion of the disturbing function for retrograde motion can be obtained from the expansion of the disturbing function for prograde motion. The two ways are equivalent as they depend on whether one chooses to invert the motion of the inner body or that of the outer one¹. Inverting the motion of the outer body gives the first transformation: $I^* = 180^\circ - I$, $\lambda'^* = -\lambda'$, $\omega^* = \omega - \pi$ and $\Omega^* = -\Omega - \pi$. In this case, the longitude of pericentre $\varpi = \omega + \Omega$ is transformed into $\varpi^* = \omega - \Omega$ and the mean longitude $\lambda =$

¹ A retrograde orbit with inclination $I > 90^\circ$ can be obtained from a prograde orbit with inclination $180^\circ - I$ by inverting the direction of motion which implies a swap between ascending and descending nodes.

$M + \omega + \Omega$ into $\lambda^* = M + \omega - \Omega$ where M is the mean anomaly. Equivalently, this means applying the generating function $F_1 = -\lambda' \Lambda'^* + (\lambda + 2z) \Lambda^* + (g - 2z) \Gamma^* - (z - \pi) Z^*$ to the usual Poincaré action-angle variables. Inverting the motion of the inner body gives the second possible transformation: $I^* = 180^\circ - I$, $f^* = -f$, $\omega^* = \pi - \omega$ and $\Omega^* = \pi + \Omega$ that may be obtained with the generating function $F_2 = \lambda' \Lambda'^* - (\lambda + 2z) \Lambda^* - (g - 2z) \Gamma^* + (z - \pi) Z^*$. We note however that these two transformations are passive in that they allow us only to obtain the expression of the resonant arguments. Once the arguments are obtained formally, the assumption that $\dot{\lambda} > 0$ and $\dot{\lambda}' > 0$ always holds. This is in contrast to the approach adopted in Paper I where an active transformation was used to study the planar dynamical problem by choosing explicitly from the outset the convention $\dot{\lambda} > 0$ and $\dot{\lambda}' < 0$.

2.2 Resonant terms

Now that we have shown how the expansion of the disturbing function for $I = 180^\circ - \beta$ is obtained from that with $I = \beta$, we may use the literal expansion of Ellis & Murray (2000) valid for prograde motion and transform the relevant resonance terms to describe the corresponding retrograde resonance.

The 2/1 retrograde resonance terms are of type $e^3 \cos(\lambda - 2\lambda' - 3\varpi)$ [term 4D3.1 with $j = 2$], and $e s^2 \cos(\lambda - 2\lambda' - \varpi + 2\Omega)$ [term 4D3.5 with $j = 2$]. The 1/2 retrograde resonance has direct and indirect terms of type $e^3 \cos(2\lambda - \lambda' - 3\varpi)$ [terms 4D3.4 with $j = 2$ and 4I3.6], and $e s^2 \cos(2\lambda - \lambda' - \varpi + 2\Omega)$ [terms 4D3.10 with $j = 2$ and 4I3.13]. The 1/1 retrograde resonance has direct and indirect terms. These are of type $e^2 \cos(\lambda - \lambda' - 2\varpi)$ [terms 4D2.1 with $j = 1$ and 4E2.2], and $s^2 \cos(\lambda - \lambda' + 2\Omega)$ [terms 4D2.4 with $j = 1$ and 4E2.6]. However, the 1/1 resonance direct terms cannot be obtained from the literal expansion of the disturbing function (since Laplace coefficients diverge when $\alpha \rightarrow 1$). We develop a semi-analytic model for the co-orbital resonance in the next section.

A similar analysis for any p/q retrograde resonance shows that there are resonant terms²

$$e^{p+q-2k} s^{2k} \cos(q\lambda - p\lambda' - (p+q-2k)\varpi + 2k\Omega). \quad (8)$$

with $k = 0, 1, 2, \dots$ and $p+q \geq 2k$.

If $\beta \ll 1$ then $s \ll 1$, hence the term with $k = 0$ is dominant. Since we restrict our study to planar retrograde resonance ($s = 0$), only the term $e^{p+q} \cos \phi$ remains, where

$$\phi = q\lambda - p\lambda' - (p+q)\varpi. \quad (9)$$

We thus recover the retrograde resonant angle from Paper I (with the expected change of sign for the term in λ'). Following Paper I, we use the notation p/q resonance when referring to a p/q retrograde resonance.

3 A model for co-orbital resonance

Consider a test particle in the co-orbital region of the secondary ($|a-1| \ll 1$). The disturbing function may be expressed using the natural angles: the fast epicyclic motion represented by the mean longitude λ of the particle and the guiding centre phase represented by the relative mean longitude $\tau = \lambda - \lambda'$. To obtain the resonance hamiltonian, the disturbing function,

² D'Alembert rule is not obeyed because the assumption $\dot{\lambda}' > 0$ and $\dot{\lambda} > 0$ with $\varpi = \omega - \Omega$ implies that we measure angles for the test particle in the opposite direction of the binary's motion.

R (Eq. 1), is averaged with respect to the fast angle λ . The corresponding function is the ponderomotive potential $S = \langle R \rangle$ used in our previous work on the co-orbital resonance (Namouni, 1999; Namouni et al., 1999). When the relative longitude τ is introduced, we may write:

$$\cos \psi = \frac{1}{2} (1 + \cos I) \cos(f - M + \tau) + \frac{1}{2} (1 - \cos I) \cos(f + M - \tau + 2\omega), \quad (10)$$

and the ponderomotive potential is given as:

$$S = \frac{1}{2\pi(1 - e^2)^{1/2}} \int_0^{2\pi} R r^2 df, \quad (11)$$

where $r = a(1 - e^2)/(1 + e \cos f)$, and the average over the mean anomaly M has been replaced by an average over f using the conservation of angular momentum. The mean anomaly M is related to the true longitude by the eccentric anomaly $\tan(E/2) = (1 - e)^{1/2}(1 + e)^{-1/2} \tan(f/2)$ and Kepler's equation $M = E - e \sin E$.

The expansion-free expression of ψ gives the natural resonant angles for planar motion. For prograde motion ($\cos I = 1$), libration occurs around $\phi = \tau$ whereas for retrograde motion ($\cos I = -1$), libration occurs around $\phi = \tau - 2\omega$ ³. Figure 1 (top row) shows the shape of the potential S as a function of the resonant argument $\phi = \tau - 2\omega$ for planar retrograde motion where $a - 1 = 0.01$. At low eccentricity, libration occurs only around $\phi = 180^\circ$ and the potential is quite shallow. This explains why in the next section we observe that low eccentricity libration orbits for relatively large mass ratios (e.g. $\mu = 0.01$) are difficult to set up as the larger the mass ratio the stronger the mutual perturbations, the more destructive the close encounters. We shall show that such librations are quite stable at smaller mass ratios. As the eccentricity is increased, the collision boundary appears and librations may occur around 0 or 180° . The extent of the libration amplitude depend on eccentricity, for $0.1 \lesssim e \lesssim 0.7$, libration around zero has the largest amplitude. Libration around 180° regain some importance as e approaches unity. As the ponderomotive potential S is derived for three-dimensional orbits, it is instructive to see how a the introduction of a small inclination modifies the dynamics as realistic orbits in the planetary three-body problem never lie exactly on the same plane. Moreover, in the co-orbital resonance, inclination is know to mitigate collisional encounters and facilitate stable orbital transitions (Namouni, 1999; Namouni et al., 1999). Figure 1 (middle and bottom rows) shows how S is modified when the retrograde orbits have a mutual inclination of 10° and ω is set to zero. As expected, collision singularities are absent. There appears a bifurcation near $e = 0.161$ where librations around 0 and 180° have comparable amplitudes and may be associated with the same energy level. The remaining features of the planar problem are present: the potential's shallowness for low eccentricity and the dominance of libration around zero for more eccentric orbits. We remark that the potential's amplitude and bifurcation modes depend on the relative semi-major axis. Figure 1 only illustrates the similarities and differences with the planar problem. We also note that in the fully three dimensional problem, the time evolution of the argument of pericenter modifies the potential's shape and equilibria whereas for planar orbits, the potential depends on the combined phase $\phi = \tau - 2\omega$. We shall present the study of the three-dimensional retrograde co-orbital resonance elsewhere.

³ Here, we define $\lambda = M + \varpi$ with $\varpi = \omega + \Omega$. If we define $\lambda = M + \varpi$ with $\varpi = \omega - \Omega$ the retrograde resonant angle is $\phi = \lambda - \lambda' - 2\varpi$ in agreement with the conclusions of the previous section.

4 Retrograde resonance phase space

Poincaré surfaces of section are useful tools to study the phase space structure in the three-body problem. In what follows, we define how we set up surfaces of section for the phase space of the 2/-1, 1/-1 and 1/-2 resonances. We then examine the types of orbits involved, as well as their potential stability.

4.1 Surface of section construction

In the barycentric rotating frame, the planar circular restricted three-body problem has two degrees of freedom (x, y) and one integral of motion, the Jacobi constant (Murray & Dermott, 1999):

$$C = x^2 + y^2 - (\dot{x}^2 + \dot{y}^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}, \quad (12)$$

where $\mu < 1$ is the mass ratio of the secondary and the primary, $r_1^2 = (x - \mu)^2 + y^2$ and $r_2^2 = (x - 1 + \mu)^2 + y^2$. Orbits therefore lie on a 3D subspace $C(x, y, \dot{x}, \dot{y}) = C$ embedded in the 4D phase space. Points of an orbit that intersects a given surface, e.g. $y = 0$, in a given direction, e.g. $\dot{y} > 0$, lie on a 2D surface of section (x, \dot{x}) . An order k resonance corresponds to a set of k islands on the surface of section (Winter & Murray, 1997a,b). Here, we prefer to define the surface of section by $\dot{x} = 0$, allowing us to follow orbit intersections in the (x, y) -plane.

We choose a mass ratio $\mu = 0.01$ that is small enough for perturbation theory to apply and keplerian osculating elements to be used. These elements (a, e, ϖ) vary on a longer scale than the orbital period. Setting $\dot{x}_0 = 0$, we have:

$$\dot{y}_0 = \pm \sqrt{x_0^2 + y_0^2 + \frac{2(1-\mu)}{|x_0 + \mu|} + \frac{2\mu}{|1 - (x_0 + \mu)|} - C}. \quad (13)$$

The transformation between the barycentric frame and astrometric frame (centered on the primary) at $t = 0$ is given as:

$$x_1 = x_0 + \mu \quad \dot{x}_1 = \dot{x}_0 - y_1 \quad (14)$$

$$y_1 = y_0 \quad \dot{y}_1 = \dot{y}_0 + x_1 \quad (15)$$

When $\dot{x}_0 = 0$ and $y_0 = 0$ we have $\dot{x}_1 = 0$ and $\dot{y}_1 = \dot{y}_0 + x_1$. Hence, the necessary and sufficient conditions for setting up prograde orbits are $x_1 > 0$ and $\dot{y}_0 > 0$ or $x_1 < 0$ and $\dot{y}_0 < 0$ (since $\dot{y}_1 > 0$ if $x_1 > 0$ and $\dot{y}_1 < 0$ if $x_1 < 0$). However, the necessary and sufficient conditions for retrograde orbits are $x_1 > 0$ and $\dot{y}_0 < -x_1$ or $x_1 < 0$ and $\dot{y}_0 > -x_1$ (since $0 > \dot{y}_0 > -x_1$ and $x_1 > 0$ implies $\dot{y}_1 > 0$, whereas $0 < \dot{y}_0 < -x_1$ and $x_1 < 0$ implies $\dot{y}_1 > 0$, i.e. these orbits are prograde). By substituting the critical value $\dot{y}_0 = -x_1$ in Eq. (13), we obtain a limit on the Jacobi constant

$$C < \frac{2(1-\mu)}{|x_0 + \mu|} + \frac{2\mu}{|x_0 - 1 + \mu|} - \mu(\mu + 2x_0), \quad (16)$$

such that, within this range of C , $x_1 > 0$ and $\dot{y}_0 < 0$ or $x_1 < 0$ and $\dot{y}_0 > 0$ are necessary and sufficient conditions for retrograde orbits. When $|x_1| < 3$, this limit is $C \lesssim 0.7$.

We construct surfaces of section defined by $\dot{x} = 0$ and $\dot{y} \times \dot{y}_0 > 0$ so that the initial condition lies on the surface of section. We vary x_0 between -3 and 3 with increments of

0.05, and $0 \leq y_0 \leq 0.8$ with increment of 0.1. The Jacobi constant $-1.9 \leq C \leq 0.7$ is incremented by 0.1. We use Eq. (13) to obtain \dot{y}_0 and choose $\dot{y}_0 > 0$ if $x_1 < 0$ and $\dot{y}_0 < 0$ if $x_1 > 0$. This ensures that the initial conditions always correspond to retrograde orbits (as $C \lesssim 0.7$). The integration time is 5×10^5 binary periods. These surfaces of section can be seen as electronic supplementary material (a selection will be discussed in Sect. 3.3).

We show below the level curves of constant C in (a, e) space for initial conditions at conjunction i.e. $x_1 > 0$ and $y = 0$ (Fig. 2 (a) and (b)) and for initial conditions at opposition i.e. $x_1 < 0$ and $y = 0$ (Fig. 2 (c) and (d)). The chosen range of C spans semi-major axes between 0.5 and 1.5 to include the 2/-1, 1/-1 and 1/-2 resonance regions.

4.2 Resonant angles

Test particles are started with $\lambda = \lambda' = 0$ (conjunction: $y = 0$ and $x + \mu > 0$) or $\lambda = 180^\circ$ and $\lambda' = 0$ (opposition: $y = 0$ and $x + \mu < 0$). The points on the surface of section ($\dot{x} = 0$) with $y = 0$ correspond to the osculating orbits' pericenter ($\lambda = \varpi$) or apocenter ($\lambda = \varpi + 180^\circ$), depending on the x and C values.

Starting in conjunction and at pericenter (apocenter) corresponds to the resonant angle $\phi = q\lambda - p\lambda' - (p+q)\varpi = 0$ ($[p+q] \times 180^\circ$). In the latter case, $\phi = 0$ (180°) if $p+q$ is even (odd). Starting at opposition and at pericenter (apocenter), $\phi = -p \times 180^\circ$ ($q \times 180^\circ$). Hence for opposition at pericenter, $\phi = 0$ (180°) if p is even (odd) and at apocenter $\phi = 0$ (180°) if q is even (odd). Therefore, even order resonant angles (such as that of the 1/-1 resonance) librate around 0 for initial conditions at conjunction, and librate around 180° for initial conditions at opposition. Odd order resonant angles with p even (such as that of the 2/-1 resonance) librate around 180° for initial conditions at apocenter, and librate around 0 for initial conditions at pericenter. Odd order resonant angles with an odd p (such as that of the 1/-2 resonance) librate around 180° for initial conditions at conjunction and apocenter, or opposition and pericenter, and librate around 0 for initial conditions at conjunction and pericenter, or opposition and apocenter.

4.3 Results of numerical integrations

4.3.1 Resonant configurations

We examine a selection of orbits in the 2/-1, 1/-1 and 1/-2 resonances shown in the frame rotating with the binary.

Fig. (3) shows 2/-1 resonant orbits. The top left panel shows an orbit with $C = +0.6$ that starts at conjunction or opposition ($y = 0$), and pericenter (mode A). The top right panel shows an orbit with $C = +0.6$ that starts at conjunction or opposition ($y = 0$) and apocenter (mode B). Resonant libration occurs around 0 in mode A and around 180° in mode B (Fig. 7). The low left panel shows an orbit with $C = +0.3$ that starts at conjunction or opposition ($y = 0$) and pericenter (mode A). The low right panel shows an orbit with $C = +0.3$ that starts at conjunction or opposition ($y = 0$) and apocenter (mode B). The latter orbit is very close to collision with the secondary.

Fig. (4) shows 1/-1 resonant orbits. The top left panel shows an orbit with $C = +0.6$ that starts at $x_1 > 0$ and $y = 0$ or $x_1 < 0$ and $y \neq 0$ (mode I). Resonant libration occurs around 0 and disruptive close encounters are avoided despite the high eccentricity (Fig. 8 left: mode I). The top right panel shows an orbit with $C = +0.6$ that starts at $x_1 < 0$ and

$y = 0$ (mode II). The 1/-1 resonant angle librates around 180° (Fig. 8 right: mode II) but the orbit is close to collision and becomes unstable when $C < 0.6$. Both orbits are described by the equilibria of the ponderomotive potential S in Fig. (1, upper row, rightmost panel).

The mid left panel of Fig. (4) shows an orbit with $C = -0.9$ and moderate eccentricity that starts at $x_1 > 0$ and $y = 0$ or $x_1 < 0$ and $y \neq 0$ (mode I). The resonant angle librates ϕ around 0. The mid right panel of Fig. (4) shows an orbit with $C = -1.1$ that is nearly circular and has initially $x_1 < 0$ and $y = 0$ (mode III). This crossing orbit is very close to collision and we expect resonant libration around 180° (Fig. 9 right: mode III). What happens is an interesting behavior best seen if we integrate a similar orbit for smaller mass ratios thus reducing the jitter due to close encounters. In Fig. (5, left panel), we plot in the (ϕ, e) -plane, a similar orbit but with $\mu = 10^{-4}$. The resonant argument alternates periodically between libration and circulation in a state that is stable over long time scales. Observing libration around $\phi = 180^\circ$ requires a finer search which becomes easier as the mass ratio is decreased (Fig. 5, right panel). The low left panel of Fig. (4) shows an orbit with $C = -1.2$ that has small eccentricity and starts at $x_1 > 0$ and $y = 0$ or $x_1 < 0$ and $y \neq 0$ (mode I). This crossing orbit is also close to collision and the resonant angle ϕ librates around 0 (Fig. 9 left: mode I). The low right panel of Fig. (4) shows an orbit with $C = -1.2$ that is nearly circular and starts at $x_1 > 0$ or $x_1 < 0$ and $y = 0$ for $C = -1.2$. This is a non-crossing orbit just exterior to the secondary's orbit and the resonant angle circulates.

Fig. (6) shows 1/-2 resonant orbits. The top left panel shows an orbit with $C = -1.5$ that starts at conjunction and pericenter, or opposition and apocenter (mode A). The top right panel shows an orbit with $C = -1.5$ that starts at conjunction and apocenter, or opposition and pericenter (mode B). Resonant libration occurs around 0 in mode A and around 180° in mode B (Fig. 10). When $C = -1.5$ both mode A and mode B orbits are close to collision with the secondary. The low left panel shows an orbit with $C = -1.2$ that starts at conjunction and pericenter, or opposition and apocenter (mode A). This is the only stable configuration when $C > -1.3$. The low right panel shows an orbit with $C = -1.8$ that starts at conjunction and apocenter, or opposition and pericenter (mode B). This is the only stable configuration when $C = -1.8$.

4.3.2 Surfaces of section

As seen in Figs. 3, 4, 6, an order k resonant orbit intersects the section $\dot{x} = 0$ at $2k$ different points. However, owing to the constraint on the sign of \dot{y} , we only see k of these intersections on the surface of section (x_1, y) . Therefore, in the surfaces of section (Fig. 11) a set of k intersections on the left hand side ($x_1 < 0$) usually represents the same configuration (in the synodic frame) as a set of k intersections on the right hand side ($x_1 > 0$).

Fig. 11 shows a selection of surfaces of section with $C = 0.6$, $C = 0.3$, $C = 0.0$, $C = -0.9$, $C = -1.1$, $C = -1.2$, $C = -1.5$ and $C = -1.8$. The full set of surfaces of section (C between 0.7 and -1.9 at steps 0.1) can be seen as electronic supplementary material (ESM). At these values of C many initial conditions correspond to crossing orbits (see Fig. 2) which can only be stable in resonance. Non-crossing small eccentricity orbits exist in the regions marked in green (left and right on Fig. 11).

When $C = 0.6$, we see nearly circular non-crossing orbits in the vicinity of the 3/-1 resonance (green) for initial conditions at opposition (left) or conjunction (right). Collision with the secondary occurs between the 3/-1 and 2/-1 resonances. We see islands of libration in 2/-1 resonance. The 2/-1 resonant orbits that start at pericenter (magenta on left and right) correspond to the same configuration in the synodic frame (Fig. 3 top left panel: mode A) where the resonant angle librates around 0 (Fig. 7 left). The 2/-1 resonant orbits that start at

apocenter (blue on left and right) also correspond to the same configuration in the synodic frame (Fig. 3 top right panel: mode B) where the resonant angle librates around 180° (Fig. 7 right).

When $C = 0.6$, we also see islands of libration in 1/-1 resonance. The 1/-1 resonant orbits that start at conjunction (black on right) or with $x_1 < 0$ and $y \neq 0$ (black on left) correspond to the same configuration in the synodic frame (Fig. 4 top left: mode I) where the resonant angle librates around 0 (Fig. 8 left). The 1/-1 resonant orbit that starts at opposition (red) is close to collision (Fig. 4 top right: mode II) and the resonant angle librates around 180° (Fig. 8 right). The 1/-1 resonance occurs at very high values of the eccentricity and the resonance center has $a = 1$.

When $C = 0.3$, we see orbits the vicinity of the 2/-1 resonance for initial conditions at opposition (left) or conjunction (right). The 2/-1 resonant angle can circulate for non-crossing orbits (green on left and right), it can librate in mode A (Fig. 3 low left) for initial conditions at pericenter (magenta on left and right), or it can librate in mode B (Fig. 3 low right) for initial conditions at apocenter (blue on left and right). Collision with the secondary occurs just outside the 2/-1 resonance separatrix. The 1/-1 resonant angle librates around 0 (mode I) for initial condition at conjunction (black on right) or with $x_1 < 0$ and $y \neq 0$ (black on left). There are also islands of libration in the 1/-2 resonance for initial condition at conjunction / pericenter (magenta on right) or opposition / apocenter (magenta on left). These correspond to the same configuration in the synodic frame where the resonant angle librates around 0 (Fig. 6 mode A).

When $C = 0.0$, we see nearly circular orbits in the vicinity of the 2/-1 resonance (green) for initial conditions at opposition (left) or conjunction (right). The 2/-1 resonant angle can only circulate and all these orbits are non-crossing. Collision with the secondary occurs between the 2/-1 and 3/-1 resonances. The 1/-1 resonant angle librates around 0 (mode I) for initial condition at conjunction (black on right) or with $x_1 < 0$ and $y \neq 0$ (black on left). The 1/-2 resonant angle librates around 0 (mode A) for initial condition at conjunction / pericenter (magenta on right) or opposition / apocenter (magenta on left).

When $C = -0.9$, all initial conditions correspond to crossing orbits hence they are only stable in resonance. The 1/-1 resonant angle librates around 0 (mode I) for initial condition at conjunction (black on right) or with $x_1 < 0$ and $y \neq 0$ (black on left). Collision with the secondary occurs in the 1/-1 resonance region. The 1/-2 resonant angle librates around 0 (mode A) for initial condition at conjunction / pericenter (magenta on right) or opposition / apocenter (magenta on left).

When $C = -1.1$, nearly circular orbits in the 1/-1 resonance starting at opposition (red on left) correspond to libration around 180° (Fig. 4 mid right: mode III) although that is not very clear from the behavior of the resonant angle (Fig. 9 right: mode III) due to the effect of repeated very close encounters. These are crossing orbits which are very close to the collision boundary. The nearly circular orbit starting at conjunction (red on right) correspond to the 1/-1 resonance separatrix. The 1/-1 resonant angle can also librate around 0 (mode I) for initial conditions at conjunction (black on left) or with $x_1 < 0$ and $y \neq 0$ (black on right). These are also crossing orbits close to the collision boundary. The 1/-2 resonant angle librates around 0 (mode A) for initial condition at conjunction / pericenter (magenta on right) or opposition / apocenter (magenta on left).

When $C = -1.2$, we see nearly circular orbits in the vicinity of the 1/-1 resonance (green) for initial conditions at opposition (left) or conjunction (right). The 1/-1 resonant angle circulates and the orbits are just exterior to the secondary's orbit thus very close to the collision boundary (Fig. 4 low right: mode III). The 1/-1 resonant angle can also librate around 0 (Fig. 9 left and Fig. 4 low left: mode I) for initial conditions at conjunction (black

on right) or with $x_1 < 0$ and $y \neq 0$ (black on left). These are crossing orbits close to the collision boundary. The 1/-1 resonance is no longer possible when $C = -1.3$ (see ESM). When $C = -1.2$, the 1/-2 resonant angle librates around 0 (mode A) for initial conditions at conjunction / pericenter (magenta on right) or opposition / apocenter (magenta on left). When $C = -1.3$ (see ESM) the 1/-2 resonant angle can also librate around 180° (mode B) for initial conditions at apocenter / conjunction and pericenter / opposition.

When $C = -1.5$, we see nearly circular orbits in the vicinity of the 2/-3 resonance (green) for initial conditions at opposition (left) or conjunction (right). Collision with the secondary occurs in the 2/-3 resonance region. There are islands of libration in 1/-2 resonance. The 1/-2 resonant orbits that start at pericenter / conjunction (magenta on right) and apocenter / opposition (magenta on left) correspond to the same configuration in the synodic frame (Fig. 6 top left panel: mode A) where resonant angle librates around 0 (Fig. 10 left). The 1/-2 resonant orbits that start at apocenter / conjunction (blue on right) and pericenter / opposition (blue on left) correspond to the same configuration in the synodic frame (Fig. 6 top right panel: mode B) where the resonant angle librates around 180° (Fig. 10 right).

When $C = -1.8$, there are nearly circular orbits in the vicinity of the 1/-2 resonance (green) for initial conditions at opposition (left) or conjunction (right). The 1/-2 resonant angle can circulate or it can librate around 180° (mode B) for initial condition at opposition / pericenter (blue on left) or conjunction / apocenter (blue on right). Collision with the secondary occurs just outside the 1/-2 resonance separatrix. The 1/-2 resonant angle can no longer librate around 0 (mode A).

4.4 Analytic model for 2/-1 and 1/-2 resonances

The structure of the 2/-1 and 1/-2 resonances, at low to moderate eccentricities, can be described by the analytic model for 3rd order resonance presented in Murray & Dermott (1999) and described in Paper I. The Hamiltonian is:

$$H = \frac{\delta}{2}(X^2 + Y^2) + \frac{1}{4}(X^2 + Y^2)^2 \mp 2X(X^2 - 3Y^2) \quad (17)$$

where δ measures the proximity to exact resonance, \mp applies to 2/-1 or 1/-2 resonances, $X = R \cos(\phi/3)$, $Y = R \sin(\phi/3)$ and R is a scaling factor specific for each resonance and dependent on the mass ratio μ (Murray & Dermott, 1999). For the 2/-1 resonance $\phi = \lambda - 2\lambda' - 3\varpi$ and $R = 2^{\frac{2}{3}}\gamma e$, where $\gamma = (3\mu f_{82})^{-1}$ and $f_{82} = 0.402$ is the amplitude of term 4D3.1 ($j = 2$) when $\alpha = 0.623$. For the 1/-2 resonance $\phi = 2\lambda - \lambda' - 3\varpi$ and $R = 2^2\gamma e$ where $\gamma = (3\mu(f_{85} - 0.5/\alpha^2))^{-1}$ where $f_{85} - 0.5/\alpha^2 = 0.533$ is the combined amplitude of the terms 4D3.4 ($j = 2$) and 4I3.6 when $\alpha = 0.623$.

A resonant orbit corresponds to a set of 3 stable equilibrium points of the Hamiltonian (Eq. 17). In Paper I we showed curves of constant Hamiltonian for several values of the parameter δ . When $\delta = 0$ (exact resonance) there is a bifurcation at the origin, and the 3 stable equilibrium points have $R = 6$ and $\phi = 0, \pm 2\pi/3$ for 2/-1 resonance and $\phi = \pi, \pm \pi/3$ for 1/-2 resonance. Applying the scaling we see that, when $\mu = 0.01$, the equilibrium points at the 2/-1 resonance have $e = 0.05$, while the equilibrium points at the 1/-2 resonance have $e = 0.03$. In Fig. 12 we show real trajectories in the vicinity of the 2/-1 resonance and in 1/-2 resonance obtained by numerical integration of the equations of motion with $\mu = 0.01$ at Jacobi constant values (C) close to bifurcation at the origin. There is very good agreement with the analytic model for the 2/-1 resonance (Fig.12a) the scaling is correct and the topology in (X, Y) coordinates is similar to the topology of the surface of section

(same equilibrium points and separatrices). However, for the 1/-2 resonance the scaling is still approximately correct but a set of 3 stable equilibrium points in the surface of section correspond to the large amplitude libration orbit in Fig.12b, possibly due to the vicinity of the separatrix at $e = 0$.

5 Discussion

This article is the continuation of our work on retrograde resonances (Paper I). We identified the transformation that must be applied to the standard expansion of the three-dimensional disturbing function in order to obtain the relevant resonance terms for retrograde motion and analyze them quantitatively.

We explored the phase-space near the retrograde resonances 1/1, 2/1, and 1/2, by constructing surfaces of section for the planar three-body problem using a mass ratio 0.01. The 1/-1 resonant term amplitude is of order e^2 and the 2/-1 and 1/-2 amplitudes are of order e^3 , hence these are the strongest retrograde resonances.

We saw that for low eccentricity non-crossing orbits, libration in the 2/-1 resonance is around 0 (corresponds to starting at conjunction or opposition and pericenter) whereas libration in the 1/-2 resonance is around 180° (corresponds to starting at conjunction and apocenter, or opposition and pericenter). These are the most stable configurations for non-crossing resonant orbits, since they ensure that closest approach with the secondary occurs always at pericenter for the 2/-1 resonance, and always at apocenter for the 1/-2 resonance. The behavior of low eccentricity 2/-1 and 1/-2 resonant orbits is in reasonable agreement with an analytic model for 3rd order resonances based on the literal expansion of the disturbing function. However, this analytic model is not valid for resonant crossing orbits. We observed that moderate to large eccentricity crossing orbits in the 2/-1 resonance or moderate eccentricity crossing orbits in the 1/-2 resonance can librate around 0 or 180° , while large eccentricity crossing orbits in the 1/-2 resonance can librate only around 0.

Recalling that the literal expansion of the disturbing function is not adequate for co-orbital motion, we developed a semi-analytic model for the 1/1 retrograde resonance based on numerical averaging of the full disturbing function and valid for large eccentricity and inclination. We saw that this model correctly explains the 1/-1 resonant modes, namely libration around 0 and around 180° . Whereas 1/-1 resonant libration around 0 is quite stable and occurs between $e \approx 0.2$ and $e \approx 1$, libration around 180° occurs only at $e \approx 0$ and $e \approx 1$, hence it is located close to the collision separatrix with the secondary. The disruptive effect of these close encounters implies that 1/-1 libration around 180° is easier to setup for smaller mass ratios ($\lesssim 10^{-4}$). We also expect that transitions between different 1/1 retrograde resonant modes are possible in the three-dimensional problem, in analogy with what is described for prograde 1/1 resonant orbits (Namouni, 1999; Namouni et al., 1999).

Acknowledgments

We acknowledge financial support from FCT-Portugal (PEst-C/CTM/LA0025/2011). The surfaces of section computations were performed on the Blafis cluster at the University of Aveiro.

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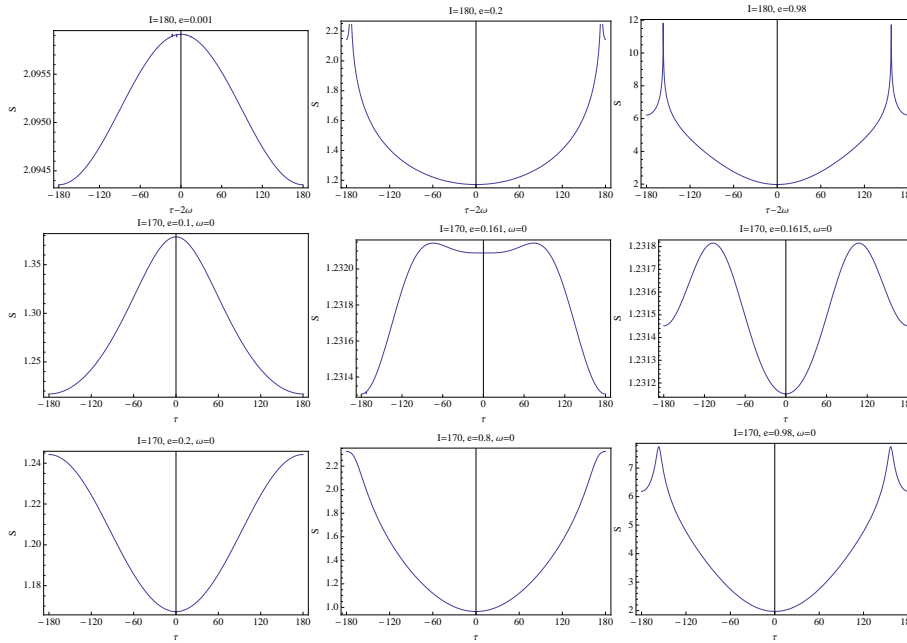


Fig. 1 Ponderomotive potential S of the 1/-1 resonance as a function of the resonant angle. The relative semi-major axis is 0.01. The upper row deal illustrates the planar potential whereas the middle and bottom rows, three dimensional orbits with small inclination. For the latter, $\tau - 2\omega$ is no longer the only possible resonant argument and S is plotted as a function of τ for a fixed ω .

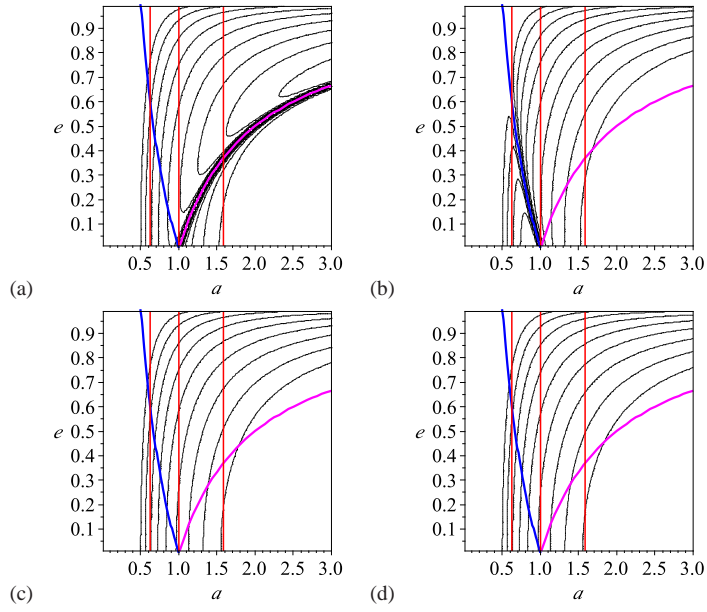


Fig. 2 Level curves of C at values (from left to right) 0.6, 0.3, 0, -0.3, -0.6, -0.9, -1.2, -1.5, -1.8. Initial condition at conjunction and pericenter (a), conjunction and apocenter (b), opposition and pericenter (c), opposition and apocenter (d). The magenta and blue lines locate collision with secondary at pericenter or apocenter. The red vertical lines show location of 2/-1, 1/-1 and 1/-2 resonances.

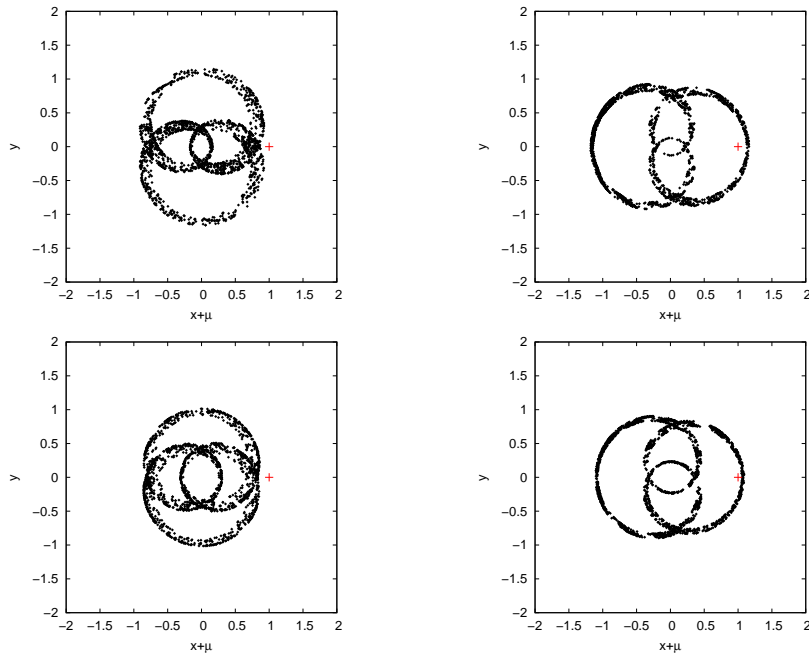


Fig. 3 Orbits in 2/-1 resonance seen in synodic frame: $C = 0.6$ mode A (top left) and mode B (top right); $C = 0.3$ mode A (low left) and mode B (low right).

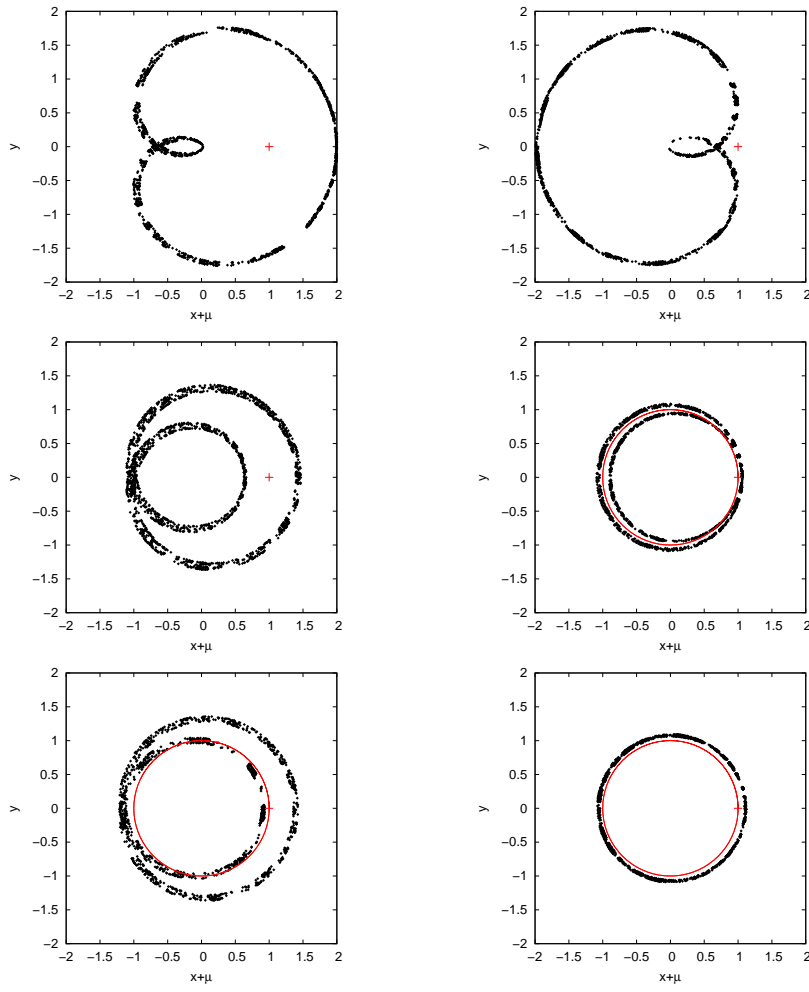


Fig. 4 Orbits in 1/-1 resonance seen in synodic frame: $C = 0.6$ mode I (top left) and mode II (top right); $C = -0.9$ mode I (mid left); $C = -1.1$ mode III libration (mid right) $C = -1.2$ mode I (low left) and mode III circulation (low right). A unit radius circle in cyan helps identify the crossing orbits and non-crossing orbits.

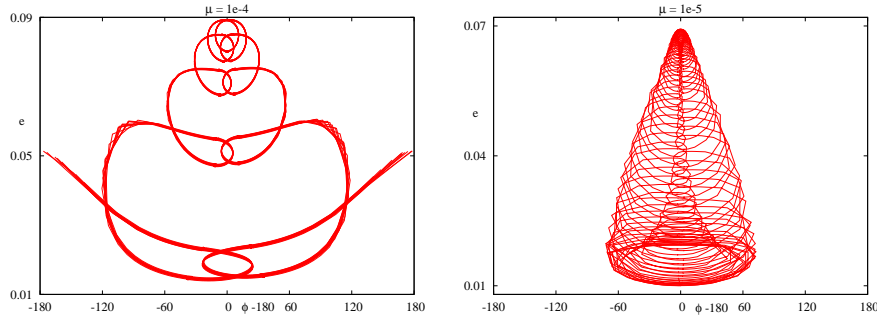


Fig. 5 Co-orbital resonant libration at small eccentricity. The orbits' initial conditions are $M = 180^\circ$, $\varpi = 0$ and for $\mu = 10^{-5}$, $a/a' - 1 = 0.001$, $e = 0.01$, whereas for $\mu = 10^{-4}$, $a/a' - 1 = 0.01$, $e = 0.08$. For better visibility, orbits are shown only for 100 periods.

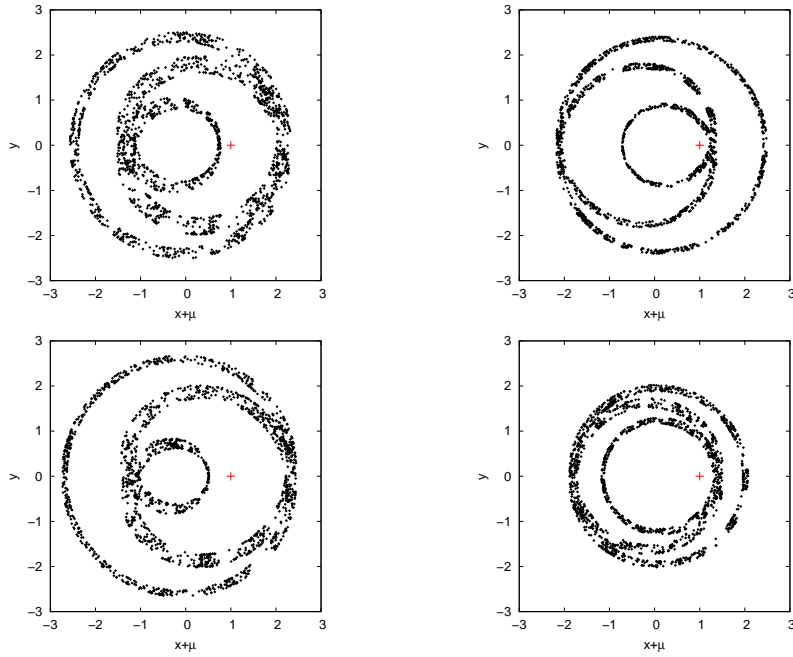


Fig. 6 Orbits in 1/2 resonance seen in synodic frame: $C = -1.5$ mode A (top left) and mode B (top right); $C = -1.2$ mode A (low left) and $C = -1.8$ mode B (low right).

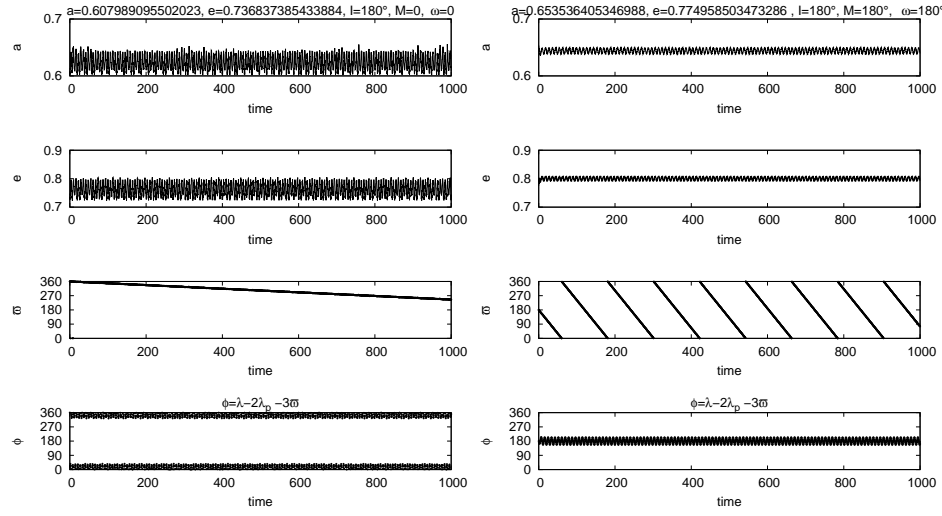


Fig. 7 Orbits in 2/-1 resonance: $C = 0.6$ mode A (left) and mode B (right).

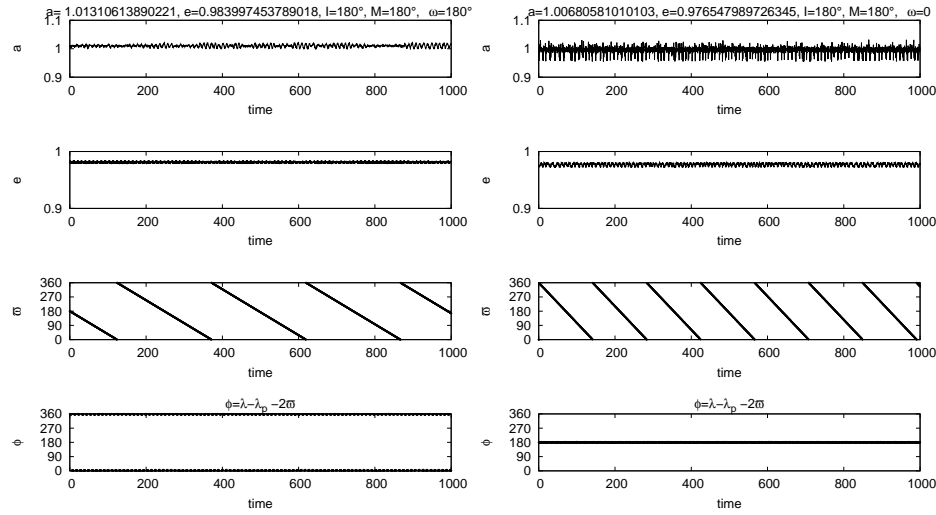


Fig. 8 Orbits in 1/-1 resonance: $C = 0.6$ mode I (left) and mode II (right).

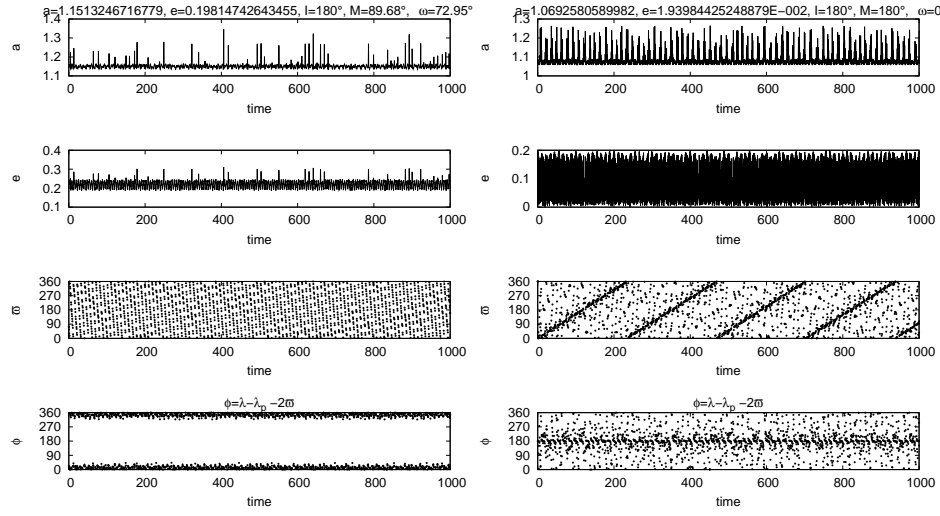


Fig. 9 Orbits in 1/-1 resonance: $C = -1.2$ mode I (left) and $C = -1.1$ mode III (right).

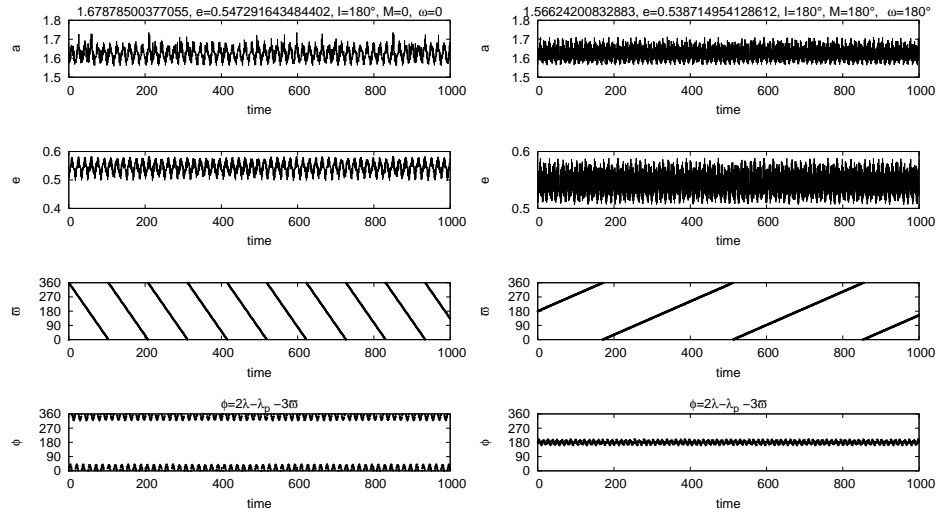


Fig. 10 Orbits in 1/-2 resonance: $C = -1.5$ mode A (left) and mode B (right).

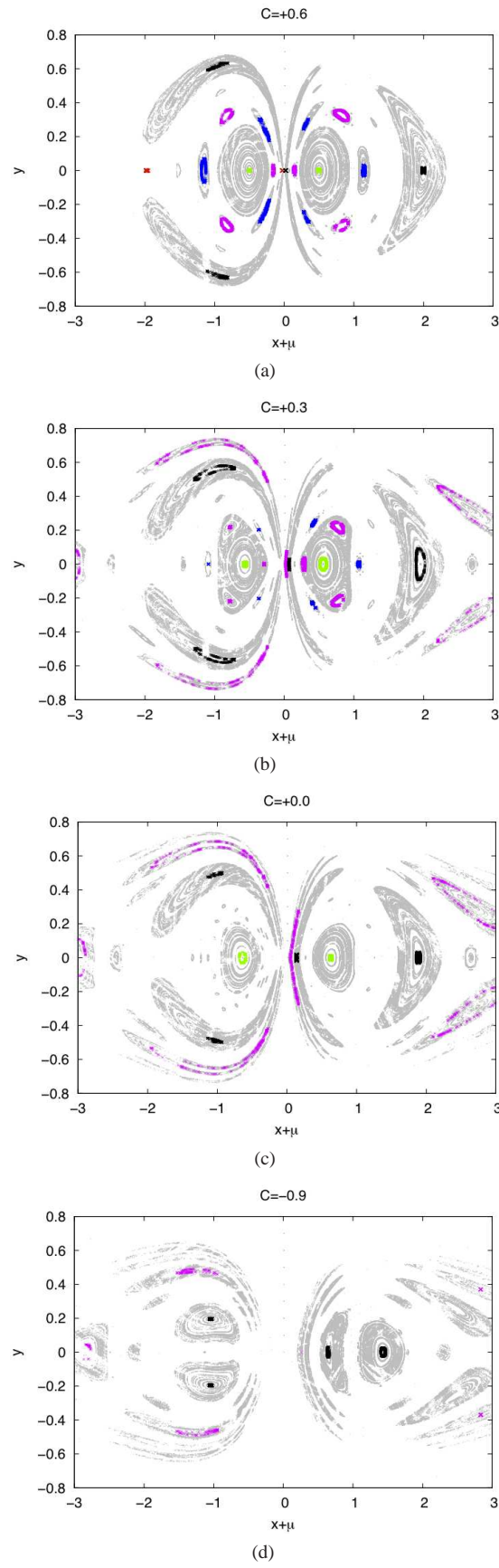


Fig. 11 Surfaces of section for selected values of C . Primary is at $(0, 0)$ and secondary is at $(1, 0)$.

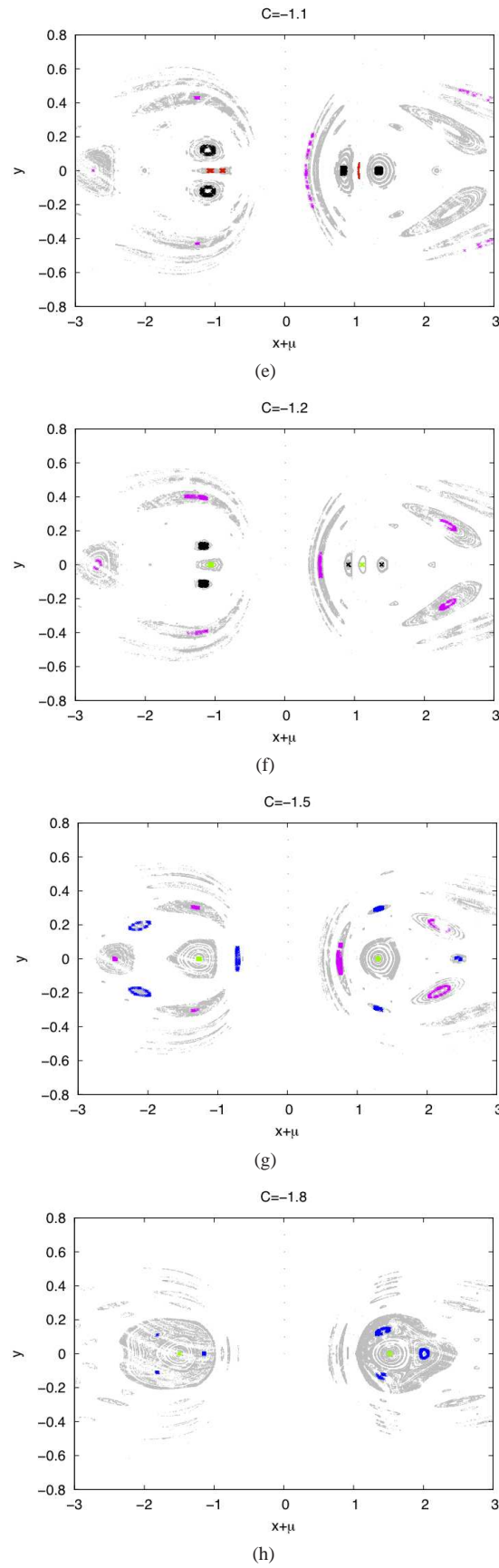


Fig. 11 Surfaces of section for selected values of C . Primary is at $(0, 0)$ and secondary is at $(1, 0)$.

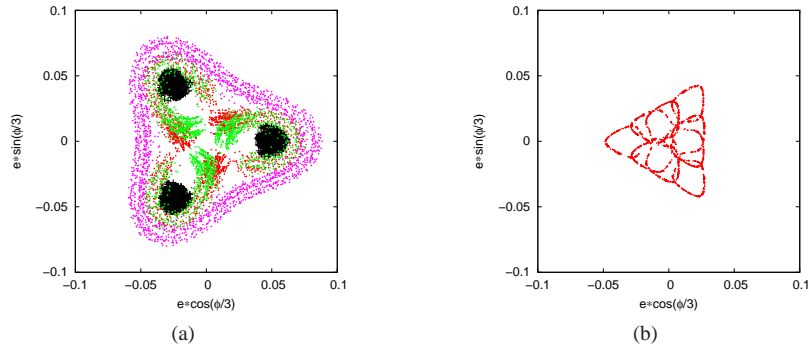


Fig. 12 (a) trajectories in the vicinity of 2/-1 resonance obtained by numerical integration of the equations of motion with $\mu = 0.01$ at $C = 0.051$: near exact resonance (black), separatrix (red and green) and outer circulation (magenta). (b) trajectories in the vicinity of 1/-2 resonance obtained by numerical integration of the equations of motion with $\mu = 0.01$ at $C = -1.886$: near exact resonance in surface of section (red).